Crab Cavity Induced Linear Synchrobetatron Coupling Resonances

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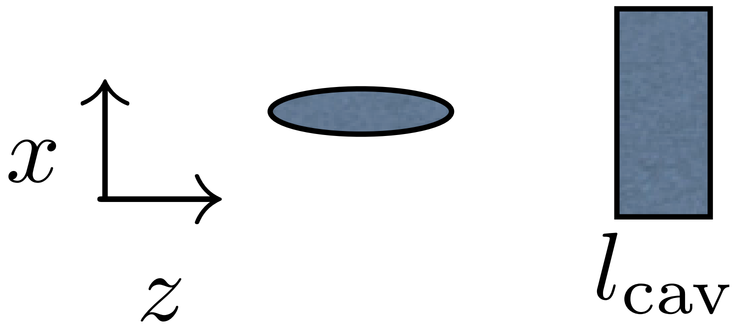
Outline

- Discussion of Crab Cavities (single vs. double)
- Coupling Effects Near Resonance for Electron Storage Rings
- Implications for LHC upgrades

Crab Cavity for Crabbed Collisions

- Crab Cavity causes tilt in x-z plane so that particles collide effectively head-on, reducing beam-beam problems from crossing angle.
- Single Crab Cavity Proposed for KEK-B
- Single vs. Double Crab cavity compensation:
  - Two crab cavities -- cost X 2
  - Enough space in IR?
  - Beam-beam tune shift partially destroys compensation.
  - Sensitive to phase errors.

Simple Crab Cavity Model



TM_{110} mode

$$\vec{z} = \begin{pmatrix} x \\ x' \\ z \\ \delta \end{pmatrix}$$

$$\Delta x' = \xi_c z$$
$$\Delta \delta = \xi_c x$$

The crab cavity map is

$$T_{\rm crab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \xi_c & 0 \\ 0 & 0 & 1 & 0 \\ \xi_c & 0 & 0 & 1 \end{pmatrix}$$

$$\xi_c = \frac{\omega_{\rm RF}}{c} \frac{qV_{\rm crab}}{E_0}$$

$$qV_{\rm crab} = c\Delta p_x$$

Hoffstaetter, Chao (2004):

$$T_{2\ \rm crab\ cav} = 1 + 4\pi\Delta\nu\beta_k \begin{pmatrix} 0 & 1 & \xi_c & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\xi_c & -\xi_c^2 & 0 \end{pmatrix}$$

“Equilibrium Beam Distribution Near Linear Synchrobetatron Coupling Resonances”

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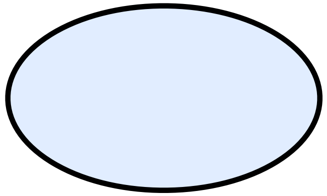
(submitted to PRST-AB)

- General Expressions for Beam Distribution in Electron Storage Ring
- Formulate perturbation theory near resonance to find approximate equilibrium beam distribution.
- Treated Crab Cavity, RF cavity dispersion

References:
Sands (1970),
A. Chao, SLIM (1979),
Ruggiero, Picasso, Radicati (1990)
Ohmi, Hirata, Oide (1994)



Electron Storage Ring with one turn map M

Find eigenvectors, construct eigen-invariants:

$$M v_a = \lambda_a v_a \qquad g_a = \vec{z}^T G_a \vec{z} \qquad G_a = -J(v_a v_a^\dagger + v_a^* v_a^T)J$$

Given the invariants for the ring and their average values, the beam distribution is

$$f(\vec{z}) = \frac{1}{\pi^3 \langle g_1 \rangle \langle g_2 \rangle \langle g_3 \rangle} \exp\left(-\frac{g_1}{\langle g_1 \rangle} - \frac{g_2}{\langle g_2 \rangle} - \frac{g_3}{\langle g_3 \rangle}\right)$$

Need to find $\langle g_a \rangle_{\mathrm{eq}}$

Beam Moments Given by

$$\Sigma = -\frac{1}{2}\langle g_1 \rangle J G_1 J - \frac{1}{2}\langle g_2 \rangle J G_2 J - \frac{1}{2}\langle g_3 \rangle J G_3 J$$

Damping and Diffusion Yield Equilibrium Invariants

Evolution equation for $\langle g_a \rangle$

$$\langle g_a \rangle (s+C) - \langle g_a \rangle (s) = -2\chi_a \langle g_a \rangle + \bar{d}_a \Longrightarrow \langle g_a \rangle_{eq} = \frac{\bar{d}_a}{2\chi_a}$$

$$\chi_a = \oint b_a \, ds \qquad \bar{d}_a = \oint d_a \, ds$$ ⟸ global

$$b_a = A_{aa} + A_{-a-a} \qquad d_a = \mathrm{Tr}[G_a D]$$

$$A = U^{-1} B U$$ ⟸ local

B and *D* are the damping and diffusion matrices and are determined by the properties of synchrotron radiation in the bending magnets and RF cavities.

To find global damping and diffusion, we need eigenvectors of the one-turn map!

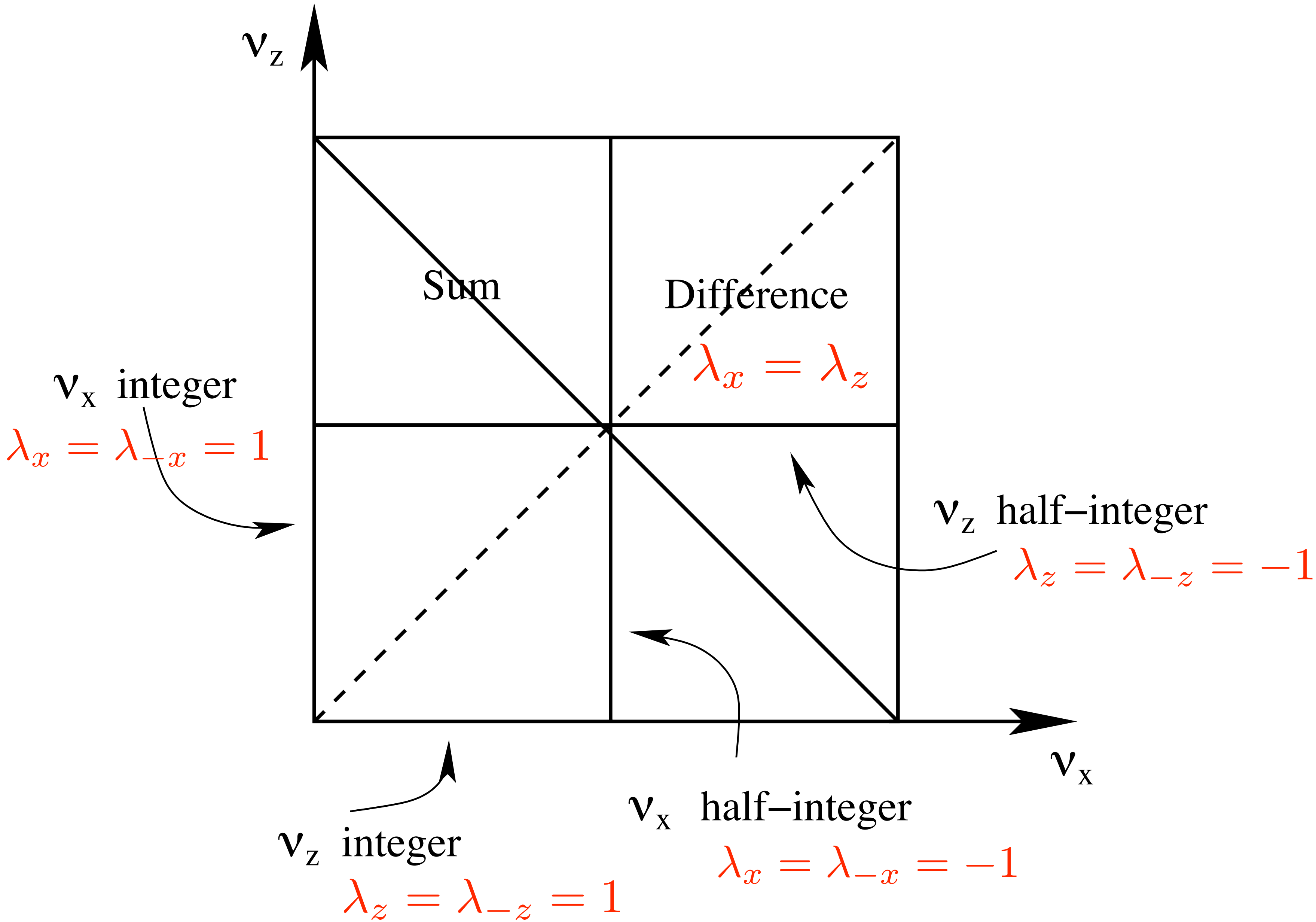
Perturbation Theory Near Resonance = Degenerate Perturbation Theory

Find Eigenvalues, Eigenvectors of One-Turn Map

$$M = (1 + P)M_0$$
$$P = \underset{\substack{\uparrow \\ \text{Difference From Resonance}}}{P_\mu} + \underset{\substack{\uparrow \\ \text{Coupling Perturbation}}}{P_\xi}$$

for multiple perturbations: $P_\xi = \sum_{m=1}^{n} T_{s_m \to C} P(s_m) T^{-1}_{0 \to s_m}$

Linear Resonances



General Perturbation Theory

$$M = M_0 + M_1 + M_2$$

$$\lambda_k = \lambda_{k0} + \lambda_{k1} + \lambda_{k2} + \dots$$

$$\tilde{v}_{k0} = \sum_j c^j_{k0} v_{k0}$$

$$\mathcal{M}_{jk} = v^{j0} M_1 v_{k0}$$

$$\mathcal{M}_{2,jk} = v^{j0} M_2 v_{k0}$$

For sum/difference and simple integer/half-integer resonances, we find:

$$\sum_j c^j_{k0} \mathcal{M}_{lj} = \lambda_{k1} c^l_{k0} \qquad l \in Z_{dg}(k)$$

For Coupling integer/half-integer we require:

$$\sum_j \left[\sum_{n \notin Z_{dg}(k)} \frac{\mathcal{M}_{nj} \mathcal{M}_{ln}}{\lambda_{k0} - \lambda_{n0}} + \mathcal{M}_{2,lj} \right] c^j_{k0} = \lambda_{k2} c^l_{k0} \quad l \in Z_{dg}(k)$$

Results...

Suppose we have degeneracy in modes (j,k)

j,k have same sign:

eigen-phase advances

$$\mu_j = \mu_{j0} + \bar{\mu} + \Delta\mu\sqrt{1 + \frac{\xi^2}{\Delta\mu^2}}$$

$$\mu_k = \mu_{k0} + \bar{\mu} - \Delta\mu\sqrt{1 + \frac{\xi^2}{\Delta\mu^2}}$$

eigen-vectors

$$\tilde{v}_{j0} = \cos\tfrac{\theta}{2} v_{j0} - ie^{-i\phi}\sin\tfrac{\theta}{2}\, v_{k0}$$

$$\tilde{v}_{k0} = -ie^{i\phi}\sin\tfrac{\theta}{2} v_{j0} + \cos\tfrac{\theta}{2} v_{k0}$$

$$\tan\theta = \frac{\xi}{\Delta\mu}$$

j,k have different sign:

$$\mu_j = \mu_{j0} + \bar{\mu} + \Delta\mu\sqrt{1 - \frac{\xi^2}{\Delta\mu^2}}$$

$$\mu_k = \mu_{k0} + \bar{\mu} - \Delta\mu\sqrt{1 - \frac{\xi^2}{\Delta\mu^2}}$$

$$\tilde{v}_{j0} = \cosh\frac{\theta}{2}\, v_{j0} - ie^{-i\phi}\sinh\frac{\theta}{2}\, v_{k0}$$

$$\tilde{v}_{k0} = ie^{i\phi}\sinh\frac{\theta}{2}\, v_{j0} + \cosh\frac{\theta}{2}\, v_{k0}$$

$$\tanh\theta = \frac{\xi}{\Delta\mu}$$

So the results are expressed in terms of xi and Delta mu for each resonance. (Also mu bar, phi).

		Difference from Resonance	Global Coupling	Local Phase	Shift in Resonance Location
reso.	condition	$\Delta\mu$ (mod 2π)	ξ	ϕ	$\bar{\mu}$
sum	$\mu_x+\mu_z=2\pi n$	$\mu_x+\mu_z-i(r_{11}-r_{-2-2})$	$2\lvert r_{1-2}\rvert$	$\arg(r_{1-2})$	$-i(r_{11}+r_{-2-2})$
diff.	$\mu_x-\mu_z=2\pi n$	$\mu_x-\mu_z-i(r_{11}-r_{22})$	$2\lvert r_{12}\rvert$	$\arg(r_{12})$	$-i(r_{11}+r_{22})$
int (x)	$\mu_x=2\pi n$	$2\mu_x-2ir_{11}$	$2\lvert r_{1-1}\rvert$	$\arg(r_{1-1})$	0
int (z)	$\mu_z=2\pi n$	$2\mu_z-2ir_{22}$	$2\lvert r_{2-2}\rvert$	$\arg(r_{2-2})$	0
$\frac{1}{2}$-int(x)	$\mu_x=\pi(2n+1)$	$2(\mu_x-\pi)-2ir_{11}$	$2\lvert r_{1-1}\rvert$	$\arg(r_{1-1})$	0
$\frac{1}{2}$-int(z)	$\mu_z=\pi(2n+1)$	$2(\mu_z-\pi)-2ir_{22}$	$2\lvert r_{2-2}\rvert$	$\arg(r_{2-2})$	0
cp. int (x)	$\mu_x=2\pi n$	$2\mu_x-2ir_{11}$ $-(\lvert r_{12}\rvert^2+\lvert r_{-12}\rvert^2)\cot(\frac{\mu_z}{2})$	$2\lvert r_{1-1}+ir_{2-1}r_{12}\cot(\frac{\mu_z}{2})\rvert$	$\arg(r_{1-1}+ir_{2-1}r_{12}\cot(\frac{\mu_z}{2}))$	0
cp. int (z)	$\mu_z=2\pi n$	$2\mu_z-2ir_{22}$ $-(\lvert r_{12}\rvert^2+\lvert r_{-12}\rvert^2)\cot(\frac{\mu_x}{2})$	$2\lvert r_{2-2}+ir_{1-2}r_{21}\cot(\frac{\mu_x}{2})\rvert$	$\arg(r_{2-2}+ir_{1-2}r_{21}\cot(\frac{\mu_x}{2}))$	0
cp. $\frac{1}{2}$-int(x)	$\mu_x=\pi(2n+1)$	$2(\mu_x-\pi)-2ir_{11}$ $+(\lvert r_{12}\rvert^2+\lvert r_{-12}\rvert^2)\tan(\frac{\mu_z}{2})$	$2\lvert r_{1-1}-ir_{2-1}r_{12}\tan(\frac{\mu_z}{2})\rvert$	$\arg(r_{1-1}-ir_{1-2}r_{21}\tan(\frac{\mu_z}{2}))$	0
cp. $\frac{1}{2}$-int(z)	$\mu_z=\pi(2n+1)$	$2(\mu_z-\pi)-2ir_{22}$ $+(\lvert r_{12}\rvert^2+\lvert r_{-12}\rvert^2)\tan(\frac{\mu_x}{2})$	$2\lvert r_{2-2}-ir_{1-2}r_{21}\tan(\frac{\mu_x}{2})\rvert$	$\arg(r_{2-2}-ir_{1-2}r_{21}\tan(\frac{\mu_x}{2}))$	0

$$\boxed{\begin{gathered} r_{jk}=v^{j0}Pv_{k0} \\ v^{j0}\equiv -i\ \mathrm{sgn}(j)\, v^{\dagger}_{j0}J \end{gathered}}$$

Matrix Elements from the perturbation.

Eigen-Invariants, damping, and diffusion coefficients

From Eigenvectors, we can find Local Invariants, and Damping and Diffusion Coefficients. Integrate to get Global quantities. Quantities satisfy sum rules, including Robinson sum rule for damping and emittance sum rules.

resonance	mode	b	b_c	χ
sum	1	$b_{x\beta}\cosh^2\frac{\theta}{2} - b_{z\beta}\sinh^2\frac{\theta}{2} + b_c\sinh\theta$		$\chi_x\cosh^2(\frac{\theta}{2}) - \chi_z\sinh^2(\frac{\theta}{2})$
	2	$-b_{x\beta}\sinh^2\frac{\theta}{2} + b_{z\beta}\cosh^2\frac{\theta}{2} - b_c\sinh\theta$	Eq.(196)	$-\chi_x\sinh^2(\frac{\theta}{2}) + \chi_z\cosh^2(\frac{\theta}{2})$
difference	1	$b_{x\beta}\cos^2\frac{\theta}{2} + b_{z\beta}\sin^2\frac{\theta}{2} + b_c\sin\theta$		$\chi_x\cos^2(\frac{\theta}{2}) + \chi_z\sin^2(\frac{\theta}{2})$
	2	$b_{x\beta}\sin^2\frac{\theta}{2} + b_{z\beta}\cos^2\frac{\theta}{2} - b_c\sin\theta$	Eq.(197)	$\chi_x\sin^2(\frac{\theta}{2}) + \chi_z\cos^2(\frac{\theta}{2})$
int/$\frac{1}{2}$-int	1	$b_{x\beta}$		χ_x
	2	$b_{z\beta}$		χ_z

resonance	mode	d	d_c	$\bar{d}$
sum	1	$d_x\cosh^2\frac{\theta}{2} + d_z\sinh^2\frac{\theta}{2} + d_c\sinh\theta$		$\cosh^2(\frac{\theta}{2})\bar{d}_x + \sinh^2(\frac{\theta}{2})\bar{d}_z$
	2	$d_x\sinh^2\frac{\theta}{2} + d_z\cosh^2\frac{\theta}{2} + d_c\sinh\theta$	$\mathrm{Tr}[G_c^+ D_\beta]$	$\sinh^2(\frac{\theta}{2})\bar{d}_x + \cosh^2(\frac{\theta}{2})\bar{d}_z$
difference	1	$d_x\cos^2\frac{\theta}{2} + d_z\sin^2\frac{\theta}{2} + d_c\sin\theta$		$\cos^2(\frac{\theta}{2})\bar{d}_x + \sin^2(\frac{\theta}{2})\bar{d}_z$
	2	$d_x\sin^2\frac{\theta}{2} + d_z\cos^2\frac{\theta}{2} - d_c\sin\theta$	$\mathrm{Tr}[G_c^- D_\beta]$	$\sin^2(\frac{\theta}{2})\bar{d}_x + \cos^2(\frac{\theta}{2})\bar{d}_z$
int/$\frac{1}{2}$-int (x)	1	$d_x\cosh\theta + d_c\sinh\theta$		$\bar{d}_x\cosh\theta$
	2	d_z	$\mathrm{Tr}[G_c^{nx} D_\beta]$	$\bar{d}_z$
int (z)	1	d_x		$\bar{d}_x$
	2	$d_z\cosh\theta + d_c\sinh\theta$	$\mathrm{Tr}[G_c^{nz} D_\beta]$	$\bar{d}_z(\cosh\theta + \cos\phi_{nz}\sinh\theta)$

Apply to Single Crab Cavity

Writing crab cavity map in betatron coordinates, we find:

$$P_{\text{crab}} = \xi_c \begin{pmatrix} -\eta & 0 & 0 & -\eta^2 \\ -2\eta' & \eta & 1 & -\eta\eta' \\ \eta\eta' & -\eta^2 & -\eta & 0 \\ 1 & 0 & 0 & \eta \end{pmatrix}$$

η , η' are dispersion and its slope at the crab cavity.

Results of P.T. applied to Crab Cavity

reso.	condition	$\Delta\mu \pmod{2\pi}$	ξ^2	ϕ	$\bar{\mu}$
sum	$\mu_x{+}\mu_z{=}2\pi n$	$\mu_x + \mu_z - \xi_c \mathcal{G}_x$	$\xi_c^2(\frac{a\beta_x}{\mu_s} - 2\eta^2)$	$\arg(\frac{a\beta_x}{\mu_s} - \eta^2 + i\eta\mathcal{G}_x)$	$\xi_c(\mathcal{G}_x - \eta\alpha_z)$
diff.	$\mu_x{-}\mu_z{=}2\pi n$	$\mu_x - \mu_z + \xi_c(\mathcal{G}_x + \alpha_z\eta)$	$\xi_c^2(\frac{a\beta_x}{\mu_s} + 2\eta^2)$	$\arg(-\eta\mathcal{G}_x - i(\frac{a\beta}{\mu_s} - \eta^2))$	$\xi_c(\mathcal{G}_x + \eta\alpha_z)$
int (x)	$\mu_x = 2\pi n$	$2\mu_x + 2\xi_c\mathcal{G}_x + \xi_c^2\frac{a\beta_x}{\mu_s^2}$	$4\beta_x\mathcal{H}_x\xi_c^2 + \frac{4a\beta_x\mathcal{G}_x}{\mu_s^2}\xi_c^3 + \frac{a^2\beta_x^2}{\mu_s^4}\xi_c^4$	π	0
int (z)	$\mu_z = 2\pi n$	$2\mu_z - \frac{1}{2}\frac{a\beta_x}{\mu_s}\cot(\frac{\mu_x}{2})\xi_c^2$	$4\eta^2\xi_c^2 + \frac{a^2\beta_x^2\cot^2(\frac{\mu_x}{2})}{4\mu_s^2}\xi_c^4$	$\arg(\frac{a\beta_x\cot(\frac{\mu_x}{2})}{4\mu_s}\xi_c^2 - i\eta\xi_c)$	0
$\frac{1}{2}$-int(x)	$\mu_x{=}\pi(2n{+}1)$	$2(\mu_x{-}\pi){+}2\xi_c\mathcal{G}_x{-}\frac{1}{4}a\beta_x\xi_c^2$	$4\beta_x\mathcal{H}_x\xi_c^2{-}a\beta_x\mathcal{G}_x\xi_c^3{+}\frac{1}{16}a^2\beta_x^2\xi_c^4$	$\arg(\frac{1}{8}a\beta_x\xi_c^2 - \mathcal{G}_x\xi_c - i\eta)$	0

We include arbitrary dispersion, assume nu_s, xi_c small.

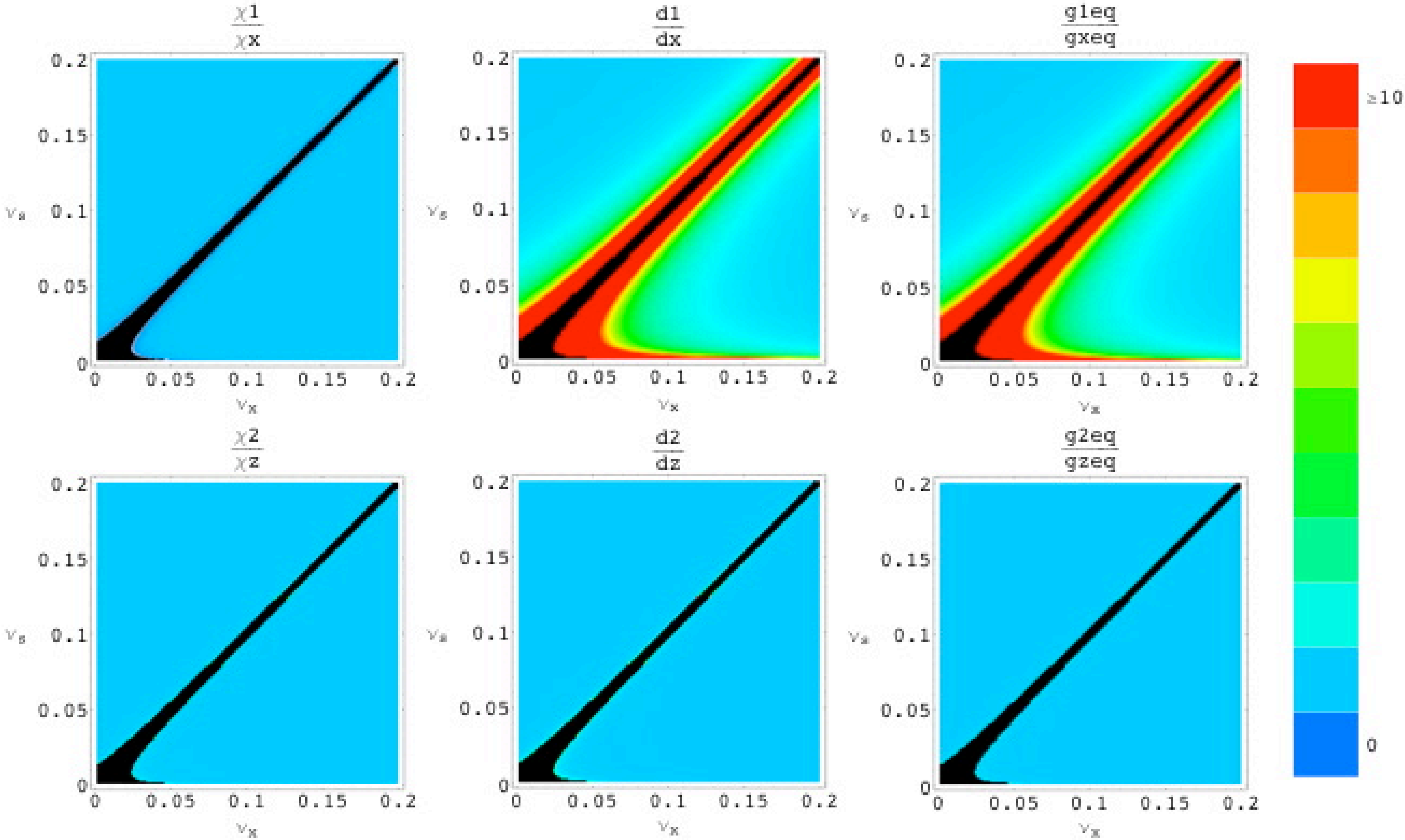
PEP II parameters

parameter	value
C	2199.33 m
α_c	1.23×10^{-3}
χ_x	1.19×10^{-4}
χ_z	2.4×10^{-4}
ϵ_x	49.2×10^{-9} m
ϵ_z	9.35×10^{-6} m
$\bar{d}_x$	2.34×10^{-11} m
d_z	8.98×10^{-9} m
$\beta(s_{\mathrm{cav}})$	20 m
$\alpha(s_{\mathrm{cav}})$	0
$\beta(s_{\mathrm{crab}})$	20 m
$\alpha(s_{\mathrm{crab}})$	0
$\eta(s_{\mathrm{cav}})$	0-3 m
$\eta'(s_{\mathrm{cav}})$	0
$\eta(s_{\mathrm{crab}})$	0-3 m
$\eta'(s_{\mathrm{crab}})$	0
ξ_c	0-.003 1/m

Sum Resonance

x

z

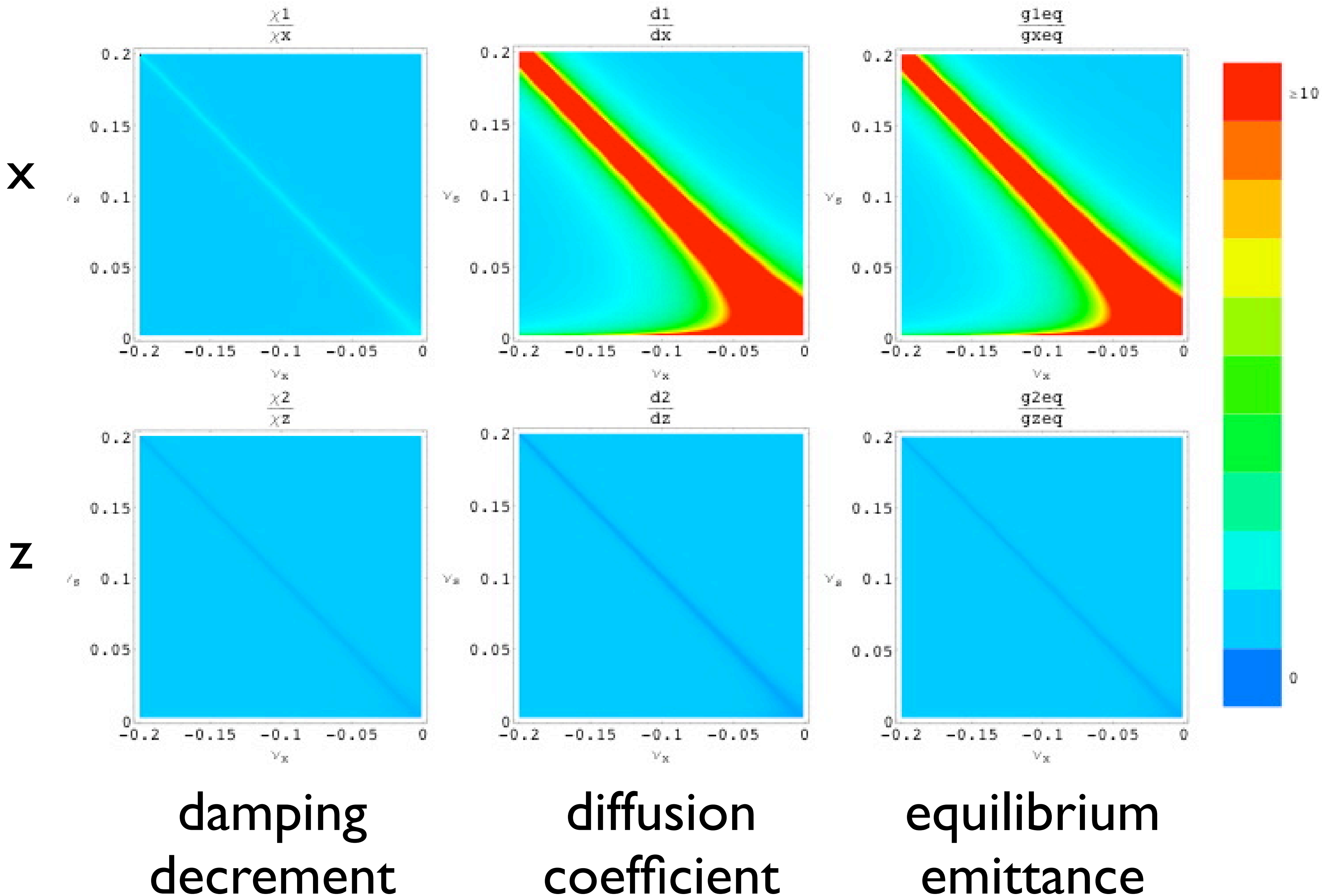


damping decrement

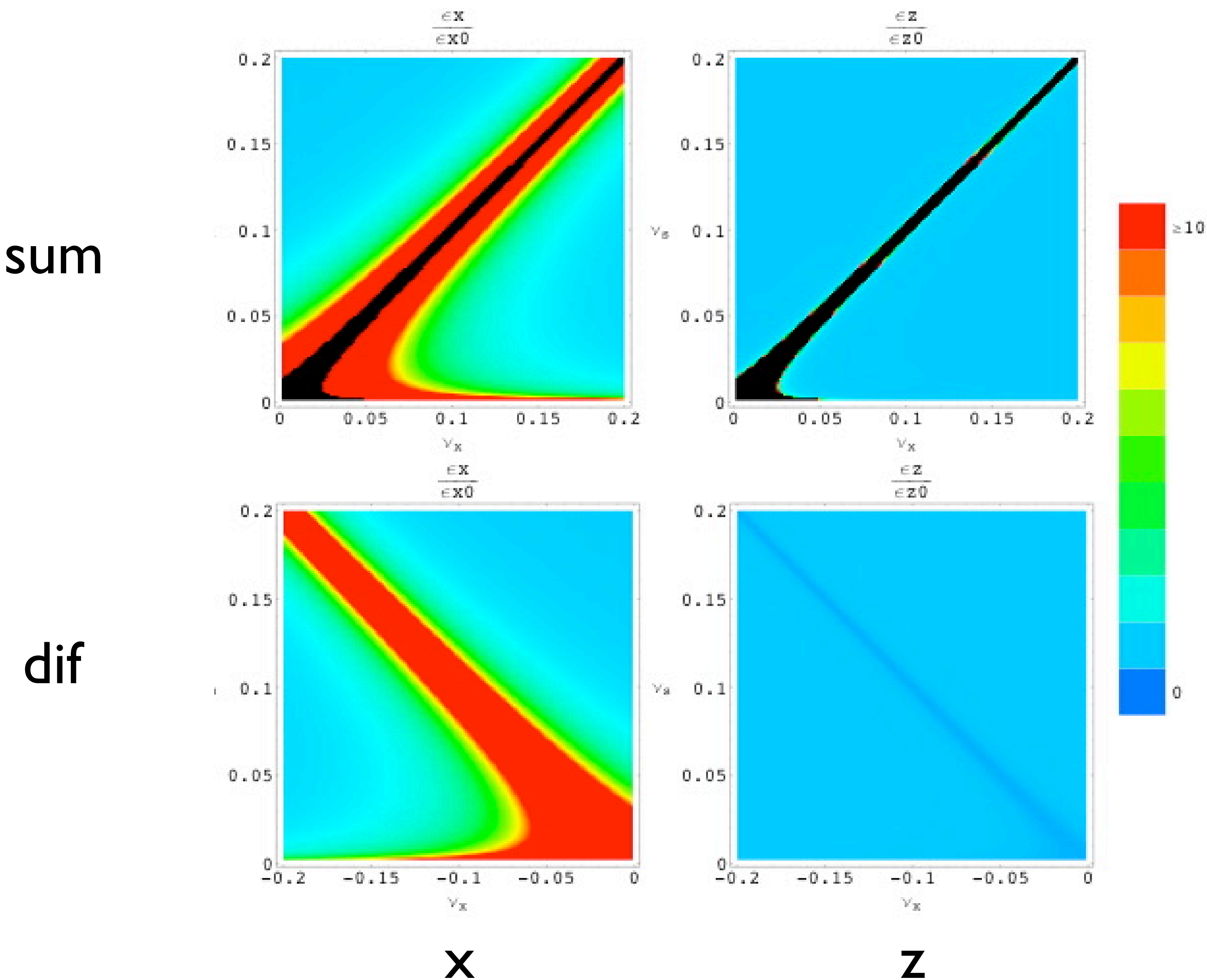
diffusion coefficient

equilibrium emittance

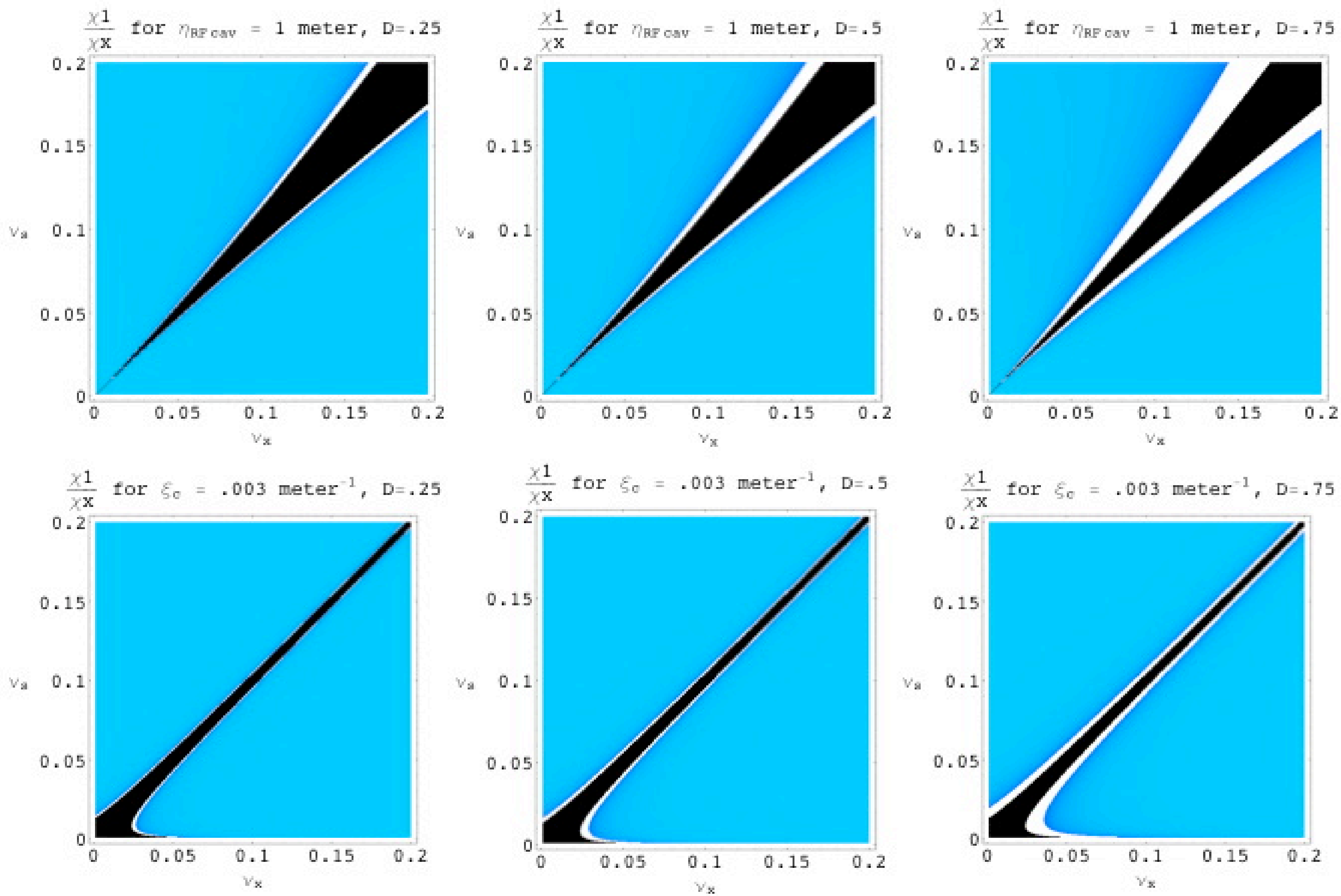
Difference Resonance



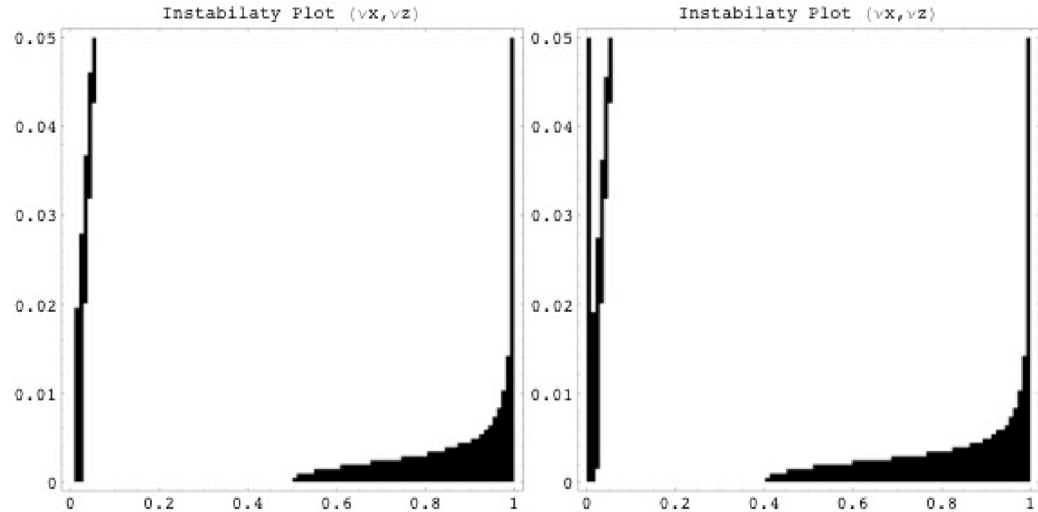
Projected Emittances



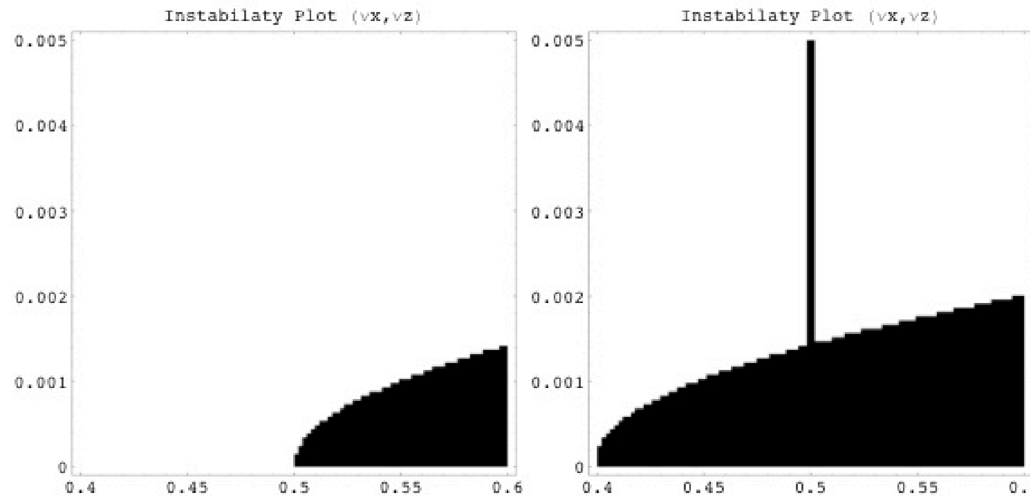
Anti-Damping Instability for Varying Damping Partition Number



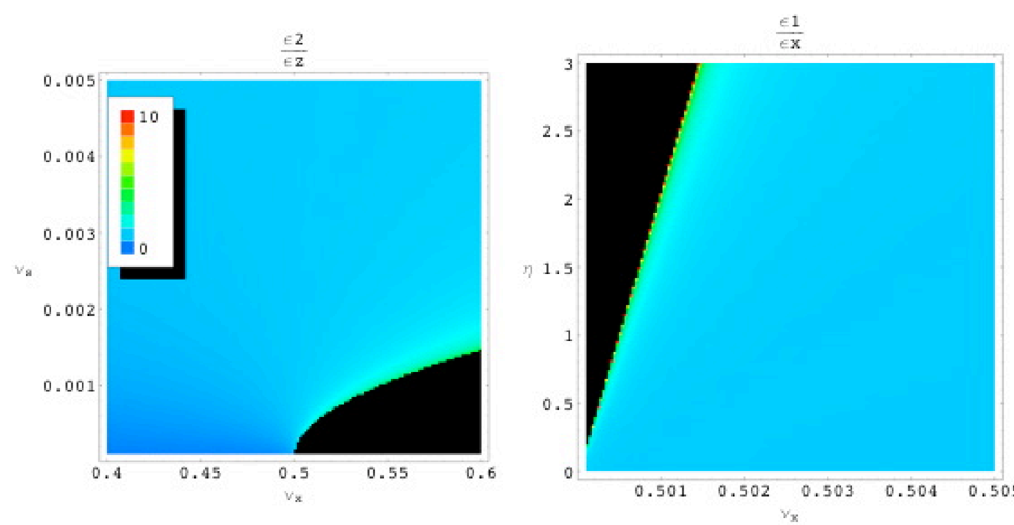
Hamiltonian Instability



magnification near half-integer



Equilibrium Transverse, longitudinal emittance



Useful Formulas

perturbed synchrotron tune:

$$\nu_2 = \sqrt{\nu_s^2 + \left(\frac{C\alpha_c\beta_x}{4}\cot(\pi\nu_x) - \eta^2\right)\left(\frac{\xi_c}{2\pi}\right)^2}$$

longitudinal emittance growth factor near integer

$$\frac{\epsilon_2}{\epsilon_z} = \frac{2\pi\nu_s + [1 + \mathrm{sgn}(\nu_x - \frac{1}{2})]\frac{C\alpha_c\beta_x}{8\pi\nu_s}\cot(\pi\nu_x)\xi_c^2}{\sqrt{(2\pi\nu_s)^2 + \frac{C\alpha_c\beta_x}{4}\cot(\pi\nu_x)\xi_c^2}}$$

horizontal emittance growth factor near half-integer ν_x:

$$\frac{\epsilon_1}{\epsilon_x} = \frac{1}{\sqrt{1 - \left[\frac{\eta\xi_c}{2\pi(\nu_x - \frac{1}{2})}\right]^2}}$$

$\eta = 1\ \mathrm{m} \quad \xi_c = 0.003\ \mathrm{m}^{-1}$

$\nu_x - \frac{1}{2} = 0.002$

$\Rightarrow \frac{\epsilon_1}{\epsilon_x} = 1.1$

General Comments

- Near half-integer safer than near integer.
- Sum/Difference resonances- very broad effect due to difference between transverse/longitudinal emittance.
- Can't avoid integer nu_s resonance! Unstable for nu_x >1/2. Strength increases near integer nu_x.

Application to LHC

- Protons instead of electrons, so not much damping, diffusion. Distribution determined by injection. Coupled invariants still relevant. Emittance growth in x due to sum/dif resonance? Beam distribution at IP?

- [nu_x] = .31 , nu_s = .003, (beta_k=20 m), integer z resonance: for xi_c = 0.003 1/m, change in nu_s ~factor of 1.5. (Bunch length decreases.)

- Parameters: C alpha_c = 8, what is xi_c, beta_k, eta_k?

- Eventually operate with nu_x above half integer because of beam-beam? If so, worry about integer nu_z instability and half-integer nu_x instability. Both are made worse by dispersion.

- Apply our formalism to crab cavity pair map?